

Salient Assemblage Slide Presentation

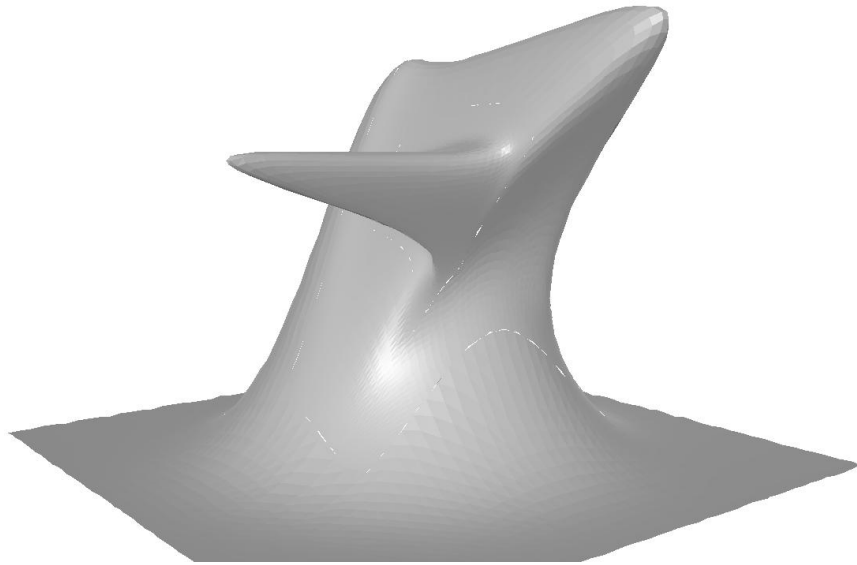
Salient Assemblage Representation
of
Multidimensional, Recursive, Deforming
Geometry

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Prototypical Salient Assemblage



Assemblage constructed from 3 *salient* units.

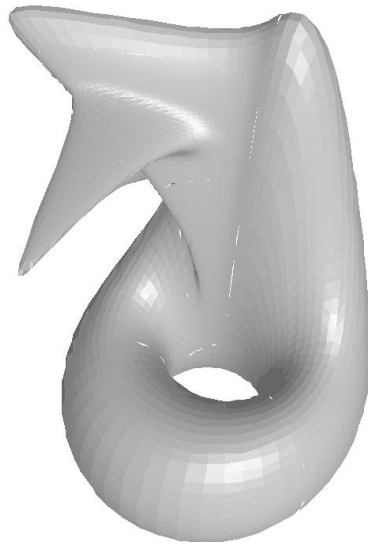
Concise data storage (24 constants)

$$\begin{bmatrix} 3.0 & 0.0 & 1.0 & 0 & 0.0 & 0.0 & 1.0 & 0 \\ 2.0 & 0.0 & 0.5 & 0 & 0.0 & 0.5 & 0.5 & 0 \\ 2.0 & 0.5 & 0.2 & 0 & 0.0 & 0.5 & 0.2 & 0 \end{bmatrix}$$

Salient Assemblage Representation $y^r = f(x^i)$ Requirements

- Add, remove, reposition, deform salient units.
- Asymptotic C^∞ salient blending.
- Topologically invariant, homeomorphic with one parameter space.
- Local control of salient direction, shape, size, and volume, at least approximately.
- Recursive attachment rules, like alignment with principal directions.
- Applies to any dimension ($i = 1, 2, \dots, n$).

Salient Assemblage is Topologically Invariant
Appended to Torus



Salient Assemblage Representation $y^r = f(x^i)$ Characteristics

- Constructive formulation, salient semiaxes form finite skeleton substructure.
- Concise data storage.
- One patch, thus no patch boundary, avoid geodesic cusp.
- Parameters usually have physical significance.
- Nowhere flat.
- Complicated algebraic expressions require computer.

Applications

- Parametric Systems (multidimensional)
 - Chemical reaction
 - Economy
 - Decision making
 - Geodesic determination
- Geometric Modeling (shape sensitive)
 - External fluid flow
 - Biological surface, deformation, growth
 - Telecommunicating complicated geometry using concise data storage

Key Issues

- What notation? Tensor notation for general curvilinear coordinate transformations.
- How to control salient direction, shape, and size, at least approximately.
- Account for parameter stretching and coordinate curve obliquity.
- Account for salient attachment in high-curvature regions.
- Efficiently compute complicated algebraic expressions.

Comparison with Other Mathematics

Frequently Asked Questions

- Why not conformal mapping? Powerful but too specialized—requires analytic mapping, preserves angle, limited to 2 dimensions, corresponds to *minimal surfaces*, a special class of manifolds. A salient has less restrictive C^∞ continuity and can be multidimensional.
- Why not 3D modeling, partition into small spline patches? Very complicated face, edge, and vertex relations in high dimensions. Patch boundaries complicate geodesic computation.
- Why not use Fourier Transform, making period arbitrarily large? Salient is more natural, not defined by an integral.
- Why is a salient a tensor-product surface? Efficient evaluation and partial derivatives, and easily extends to higher dimensions.
- Can a salient be a minimal surface? No. It has non-constant curvature. It is nowhere flat.

Comparison with Spline Representations

	Assemblage	Multi-patch Splines
primitive	salient	spline
formulation	function	discrete
recursive	yes	no
topology modeling	invariant	flexible
parameters	physical	arbitrary
patch coverage	large	small
patch boundary	C^∞	C^2
data storage	salient constants	control vertices

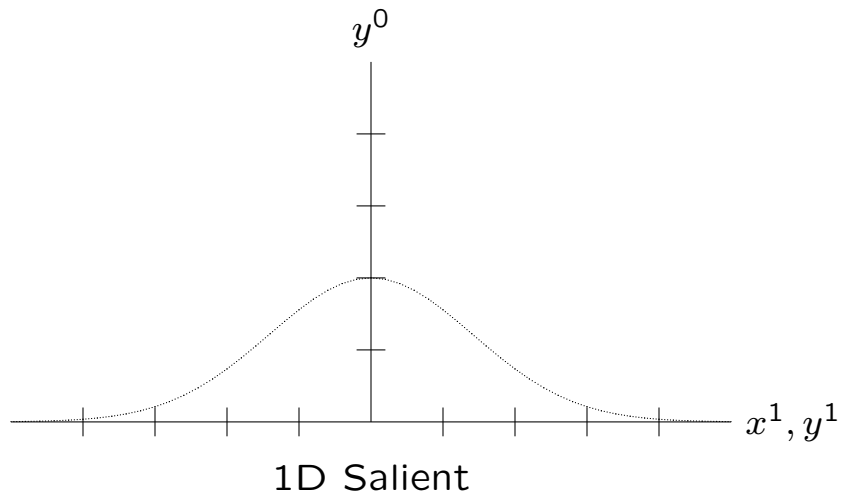
Both are parametric representations and are compatible.

Presentation Overview

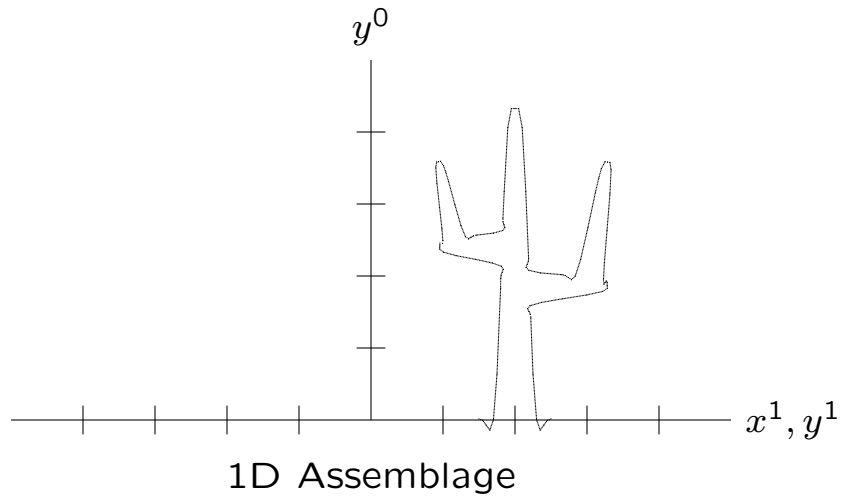
1. Describe a salient.
2. Describe ExpHermite salient, a generalized Fourier series.
3. Describe salient attachment rules.
4. Derive parametric representation $y^r = f(x^i)$.
5. Apply differential geometry methods, e.g. geodesics.

Definitions

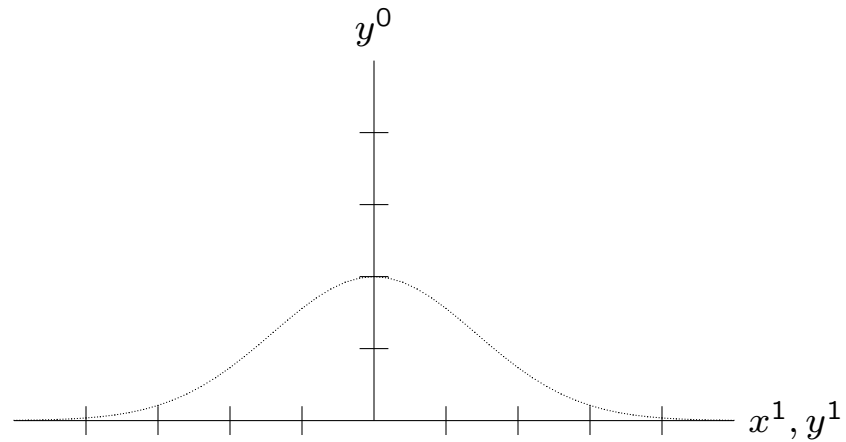
Definition 1 A **salient** is the mathematical representation of a distinguishable geometric part. It is a class C^∞ bounded function on \mathbb{R} that, along with all its bounded derivatives, vanishes sufficiently far from one set of parametric arguments.



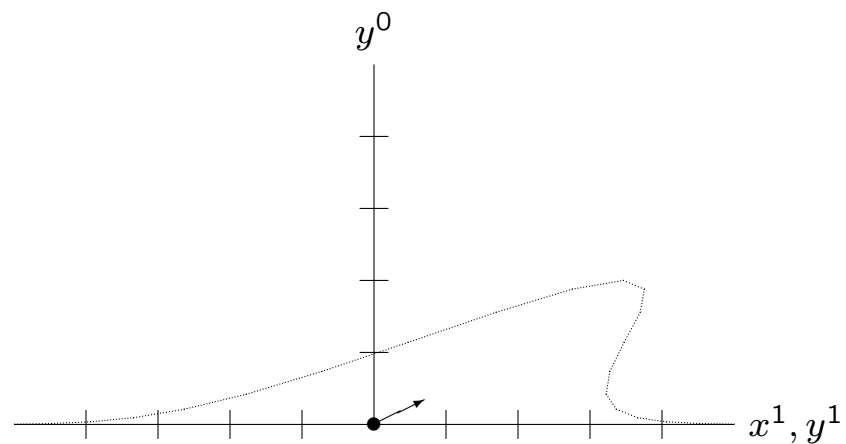
Definition 2 An **assemblage** is a collection of attached salients.



1D Salient



1D Salient

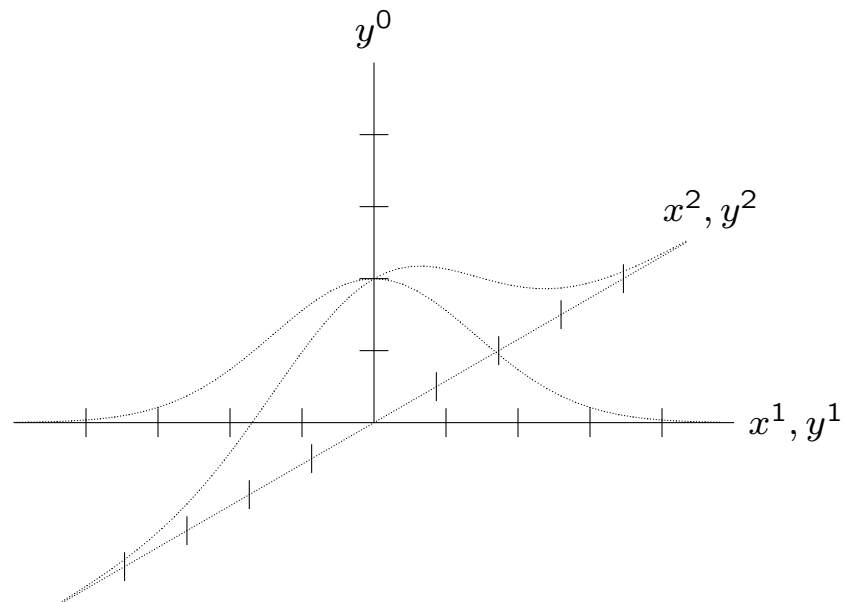


Main Semiaxis Direction (Dihedral)

$$\begin{aligned}y^0 &= \eta^0 \hat{S}, \\y^1 &= x^1 + \eta^1 \hat{S},\end{aligned}$$

where η^r are direction cosines.

2D Salient



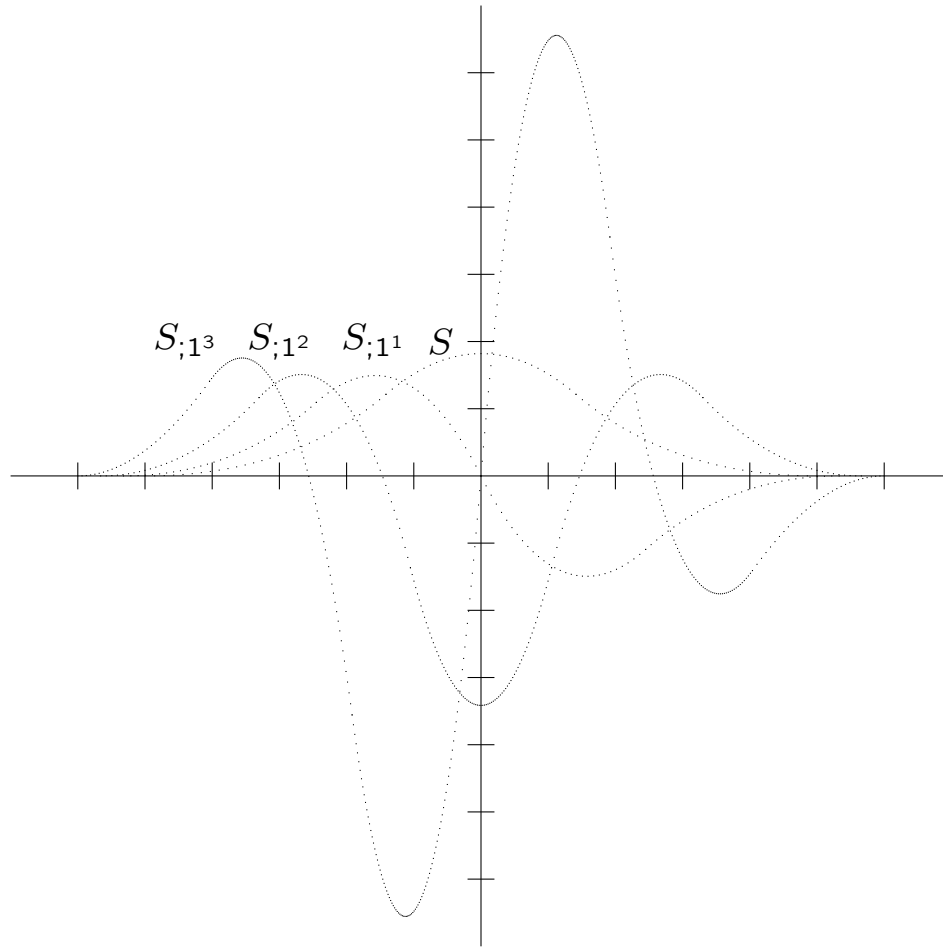
$$\begin{aligned}y^0 &= \eta^0 \hat{S}, \\y^1 &= x^1 + \eta^1 \hat{S}, \\y^2 &= x^2 + \eta^2 \hat{S}.\end{aligned}$$

More concise notation for any dimension

$$y^r = \delta_i^r x^i + \eta^r \hat{S},$$

where $r = 0, 1, \dots, n; i = 1, 2, \dots, n$
and $\eta^\rho \eta^\rho = 1$.

Salient Derivatives are Salients



1D Salient and Its First Three Derivatives

If 1D salient S and its derivatives are linearly independent, then linear combination

$$\begin{aligned}\hat{S} &= c^0 S + c^1 S_{;1} + c^2 S_{;11} + \dots + c^{n_h} S_{;1^{n_h}} \\ &= c^h S_{;1^h} \quad (\text{sum on } h = 0, 1, \dots, n_h).\end{aligned}$$

spans a wider collection of 1D salients.

2D Salient

A linear combination of a 2D salient $S(x^1, x^2)$ and its derivatives

$$\hat{S} = c^{h_1 h_2} S_{;1^{h_1} 2^{h_2}}(x^1, x^2),$$

is also a 2D salient.

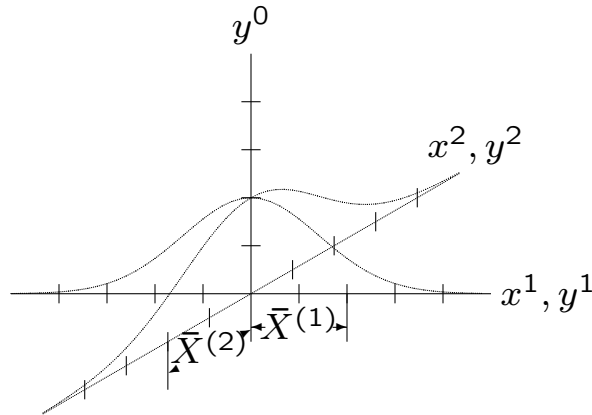
Consider only factorable S . Then \hat{S} is a *tensor-product surface*,

$$\begin{aligned} \hat{S} &= \left(c_{(1)}^{h_1} S_{(1);1^{h_1}}(\bar{x}^1) \right) \cdots \left(c_{(n)}^{h_n} S_{(n);n^{h_n}}(\bar{x}^n) \right), \\ &= \prod_j c_{(j)}^{h_j} S_{(j);j^{h_j}}(\bar{x}^j). \end{aligned}$$

Consequently,

$$y^r = \delta_i^r x^i + \eta^r \hat{S} = \delta_i^r x^i + \eta^r \prod_j c_{(j)}^{h_j} S_{(j);j^{h_j}}(\bar{x}^j).$$

Salient Nomenclature



Although a salient is open and unbounded, ellipse nomenclature is useful.

Definition 3 *Salient origin*, denoted by \dot{X}^i , is the salient's local coordinate origin.

Local curvilinear coordinates, centered on salient origin, are

$$\bar{x}^i \equiv x^i - \dot{X}^i.$$

Definition 4 *Salient main semiaxis* is the line segment from salient origin in direction η^r .

Definition 5 *Salient height* is main semiaxis length.

Definition 6 *Salient vertex* is main semiaxis endpoint.

Definition 7 *Salient \bar{x}^j -semiaxis* is the positive canonical coordinate \bar{x}^j axis.

Definition 8 *Salient semiaxis width* $\bar{X}^{(j)}$ is the \bar{x}^j -semiaxis radial width at which salient height is $1/e$ times the main semiaxis height.

Local Curvilinear to Canonical Coordinate Transformation

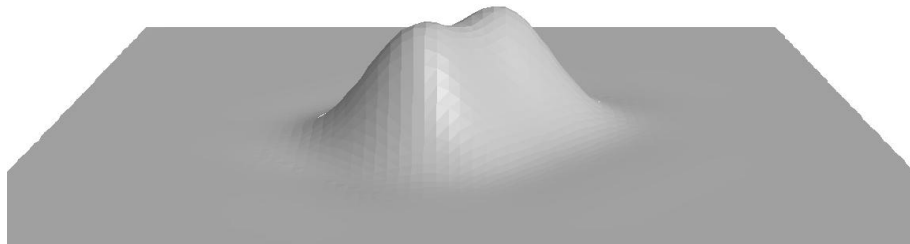
In two-dimensions, scaling and rotation transformations are

$$\begin{bmatrix} \bar{x}^1 \\ \bar{x}^2 \end{bmatrix} = \begin{bmatrix} 1/\bar{X}^{(1)} & 0 \\ 0 & 1/\bar{X}^{(2)} \end{bmatrix} \begin{bmatrix} \zeta_1^1 & \zeta_2^1 \\ \zeta_1^2 & \zeta_2^2 \end{bmatrix} \begin{bmatrix} \bar{x}^1 \\ \bar{x}^2 \end{bmatrix}.$$

In any dimension,

$$\bar{x}^j = \chi_\alpha^j \zeta_i^\alpha \bar{x}^i.$$

2D Salient (Tensor-Product Surface) with Two Shapes



Rectangle and Cone Approximations (5 terms)

In this case, salient semi-axes are rotated $\pi/4$ from rectangular axes.

Candidate Salient Functions

- $\exp(-x^2)$
- $2 \exp(x) / (1 + \exp(2x))$
- $\sin(ax)/x$
- Bessel function $J_0(x)$
- $J_1(x)/x$
- $\operatorname{sech}(x)$
- $1/(1 + ax^2)$

Exponent-Salient Function

$$\exp\left(-\left((x^1)^2 + \dots + (x^n)^2\right)\right) \equiv \exp\left(-x^i x^i\right).$$

continuous for parametric arguments but negligible sufficiently far from origin $(x^1, x^2, \dots, x^n) = (0, 0, \dots, 0)$.

Approximate values are:

x^1	$\exp\left(-\left(x^1\right)^2\right)$
0	1
1	0.36788
2	0.01831
3	1.23410×10^{-4}
4	1.12535×10^{-7}
5	1.38879×10^{-11}

Hermite Polynomials

Exponent-salient function has derivatives of all orders

$$\frac{d^h}{dx^h} \exp(-x^2) = \exp(-x^2) H_h(x).$$

Definition 9 Hermite polynomials are

$$H_0(x) = 1,$$

$$H_1(x) = -2x,$$

$$H_{h+1}(x) = -2(xH_h(x) + hH_{h-1}(x)).$$

First few Hermite Polynomials

$$H_0(x) = 1,$$

$$H_1(x) = -2x,$$

$$H_2(x) = -2 + 4x^2,$$

$$H_3(x) = 12x - 8x^3,$$

$$H_4(x) = 12 - 48x^2 + 16x^4,$$

$$H_5(x) = -120x + 160x^3 - 32x^5.$$

ExpHermite Series

Hermite polynomial products, weighted by $\exp(-x^2)$, are orthogonal,

$$\int_{-\infty}^{\infty} \exp(-x^2) H_h(x) H_\theta(x) dx = \begin{cases} 0 & \text{if } h \neq \theta \\ 2^h h! \sqrt{\pi} & \text{if } h = \theta. \end{cases}$$

Expand given salient function as an *ExpHermite series*

$$f(x) = \exp(-x^2) \sum_{h=0,1,\dots,n_h} c^h H_h(x) \quad (\text{sum on } h = 0, 1, \dots, n_h).$$

where c^h are *ExpHermite coefficients* and the ExpHermite series is a *generalized Fourier series*. To find c^h , multiply both sides by $H_\theta(x)$,

$$f(x)H_\theta(x) = \exp(-x^2) \sum_{h=0,1,\dots,n_h} c^h H_h(x) H_\theta(x).$$

Integrating both sides gives

$$\int_{-\infty}^{\infty} f(x)H_\theta(x) dx = \sum_{h=0,1,\dots,n_h} c^h \int_{-\infty}^{\infty} \exp(-x^2) H_h(x) H_\theta(x) dx.$$

Because of orthogonality, for any particular h ,

$$\int_{-\infty}^{\infty} f(x)H_h(x) dx = c^h \int_{-\infty}^{\infty} \exp(-x^2) (H_h(x))^2 dx.$$

From first equation above,

$$c^h = \frac{1}{2^h h! \sqrt{\pi}} \int_{-\infty}^{\infty} f(x)H_h(x) dx.$$

ExpHermite Coefficients for Special Shapes

Using

$$c^h = \frac{1}{2^h h! \sqrt{\pi}} \int_{-\infty}^{\infty} f(x) H_h(x) dx,$$

determine coefficients:

	$f(x)$	c^0	c^2	c^4	c^6
exponent	$\exp(-x^2)$	1	0	0	0
rectangle	1	$\frac{2}{\sqrt{\pi}}$	$\frac{-1}{6\sqrt{\pi}}$	$\frac{-1}{240\sqrt{\pi}}$	$\frac{29}{20160\sqrt{\pi}}$
cone	$1 - x $	$\frac{1}{\sqrt{\pi}}$	$\frac{-1}{6\sqrt{\pi}}$	$\frac{19}{1440\sqrt{\pi}}$	$\frac{-13}{20160\sqrt{\pi}}$
parabola	$1 - x^2$	$\frac{4}{3\sqrt{\pi}}$	$\frac{-1}{5\sqrt{\pi}}$	$\frac{11}{840\sqrt{\pi}}$	$\frac{-37}{90720\sqrt{\pi}}$
semicircle	$\sqrt{1 - x^2}$	$\frac{\sqrt{\pi}}{2}$	$\frac{-\sqrt{\pi}}{16}$	$\frac{\sqrt{\pi}}{384}$	$\frac{\sqrt{\pi}}{18432}$

These shapes are even functions with unit height and unit semiaxis width.

Approximation by ExpHermite Series

Rectangle

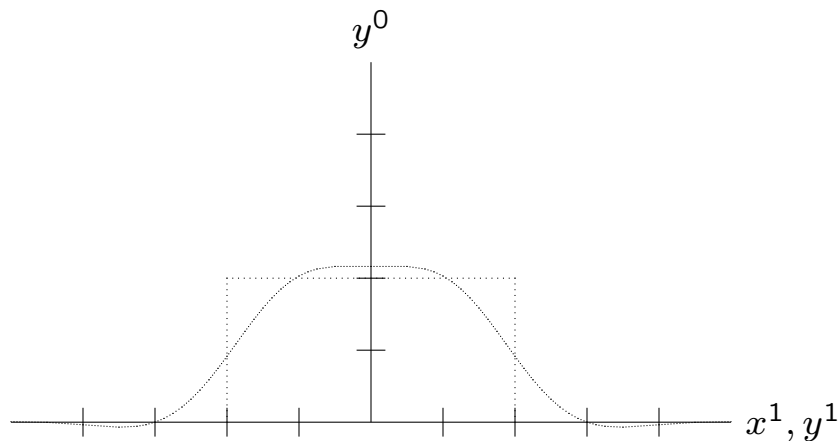
$$f(x) = \begin{cases} 0 & \text{if } x < -1 \\ 1 & \text{if } -1 \leq x \leq 1 \\ 0 & \text{if } x > 1, \end{cases}$$

is approximated by

$$f(x) \approx \frac{\exp(-x^2)}{\sqrt{\pi}} \left(2 - \frac{1}{6}H_2(x) - \frac{1}{240}H_4(x) + \frac{29}{20160}H_6(x) - \frac{67}{580608}H_8(x) \right).$$

To compute, transform to power series

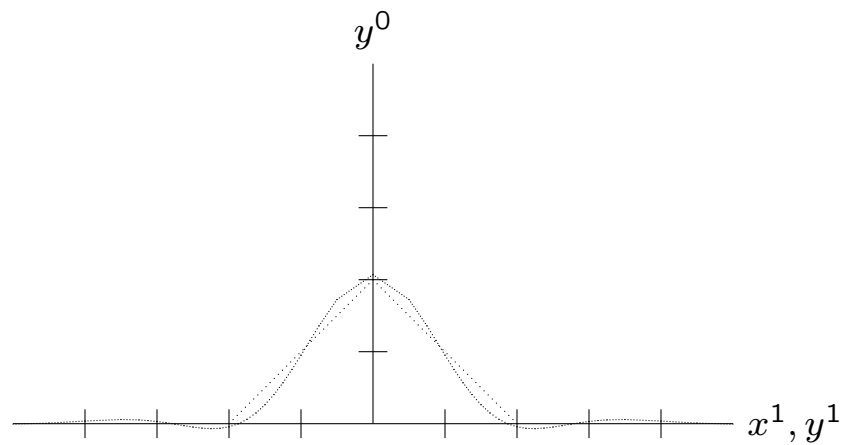
$$f(x) \approx \exp(-x^2) \left(((((-0.01667x^2 + 0.28528)x^2 - 1.30219)x^2 + 1.19607)x^2 + 1.08147) \right).$$



Approximation by ExpHermite Series

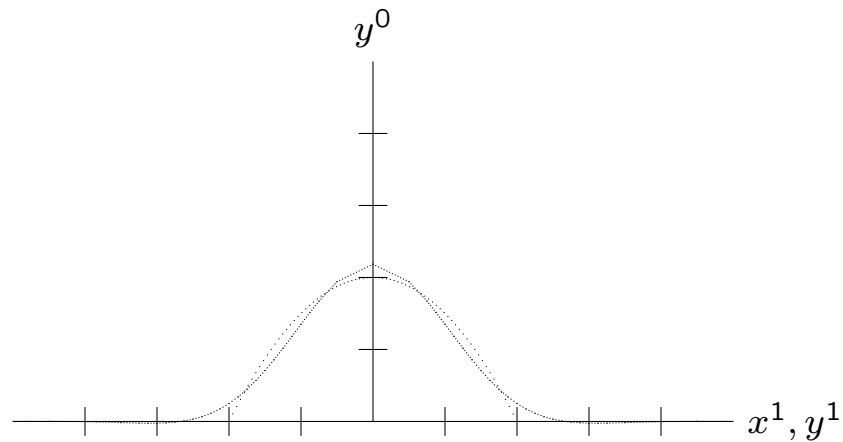
Cone approximation (5 terms)

$$f(x) = \begin{cases} 0 & \text{if } x < -1 \\ 1 - |x| & \text{if } -1 \leq x \leq 1 \\ 0 & \text{if } x > 1. \end{cases}$$

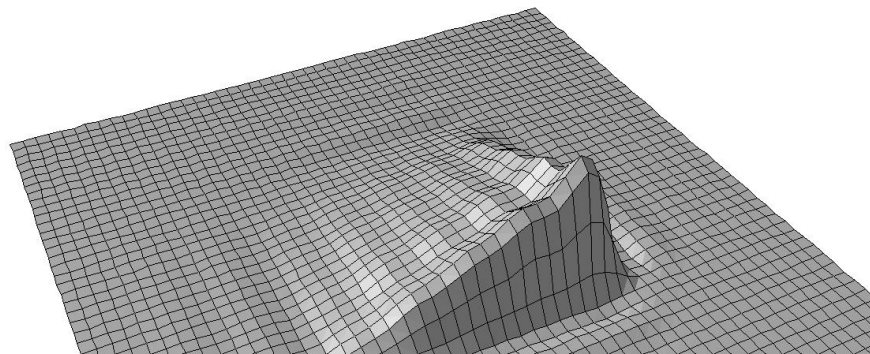


Parabola approximation (5 terms)

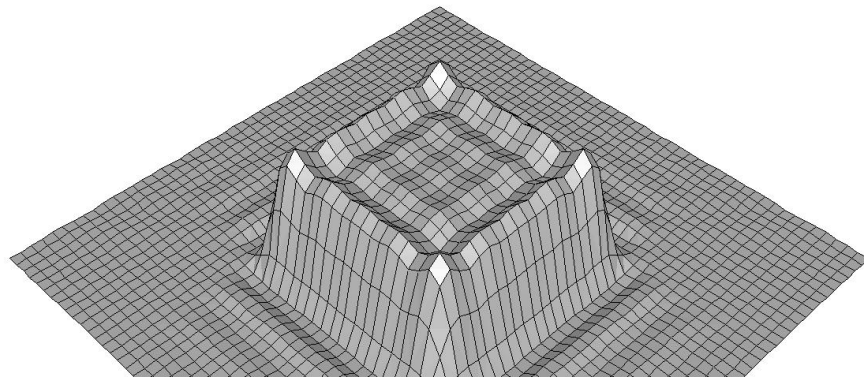
$$f(x) = \begin{cases} 0 & \text{if } x < -1 \\ 1 - x^2 & \text{if } -1 \leq x \leq 1 \\ 0 & \text{if } x > 1. \end{cases}$$



2D Ramp Approximation by Tensor-Product of ExpHermite Series



2D Rectangle Approximation by Tensor-Product of ExpHermite Series



ExpHermite Series Successive Approximations Change Shape but not Volume

Since

$$\int_{-\infty}^{\infty} \exp(-x^2) dx = \sqrt{\pi},$$

and for $h > 0$,

$$\int_{-\infty}^{\infty} \exp(-x^2) H_h(x) dx = 0,$$

then volume V under approximating surface is

$$\begin{aligned} V &= \int_{\mathcal{X}} \exp(-\bar{x}^\gamma \bar{x}^\gamma) \prod_j c_{(j)}^{h_j} H_{h_j}(\bar{x}^j) dx^1 \dots dx^n, \\ &= \left(\prod_j c_{(j)}^0 \right) \int_{\bar{\mathcal{X}}} \exp(-\bar{x}^\gamma \bar{x}^\gamma) d\bar{x}^1 \dots d\bar{x}^n, \\ &= \pi^{n/2} \prod_j c_{(j)}^0. \end{aligned}$$

Assemblage Definitions

Definition 10 *An **assemblage** is a collection of attached salients.*

Definition 11 *A salient's **parent** is the assemblage to which it is attached.*

Definition 12 *A salient is a **child** to its parent.*

Definition 13 *A child's **bud** is the point $\dot{Y}^r = y^r(\dot{X}^i)$, located on the parent.*

Definition 14 *A child's **dihedral** is the minimum angle its main semi-axis forms with the parent's tangent plane at the bud.*

Salient Attachment by Vector Addition

bark

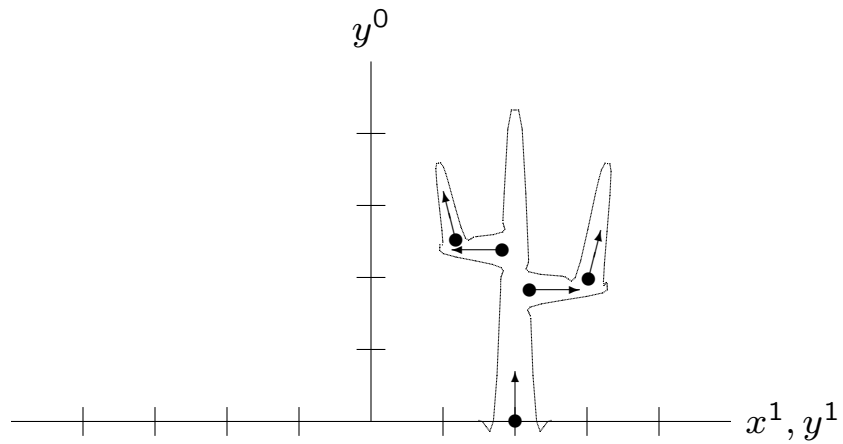
bark-bud

bark-bud-branch

bark-bud-branch

bark-bud-branch

Each salient depends on all its parents.

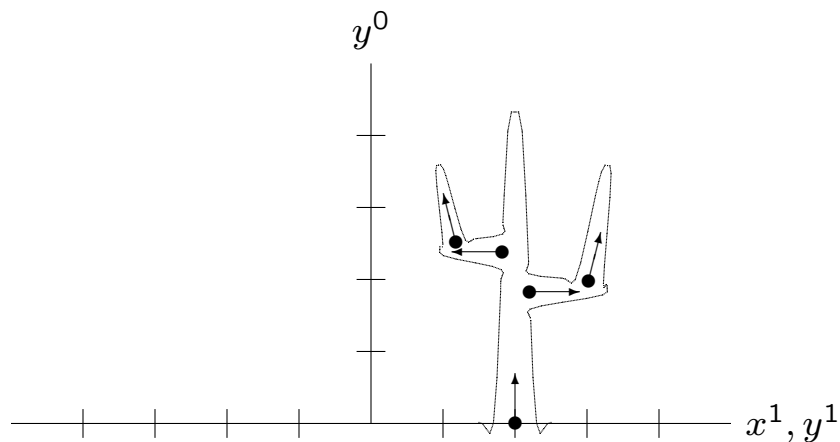


Salient Direction Cosine (Dihedral) Rule

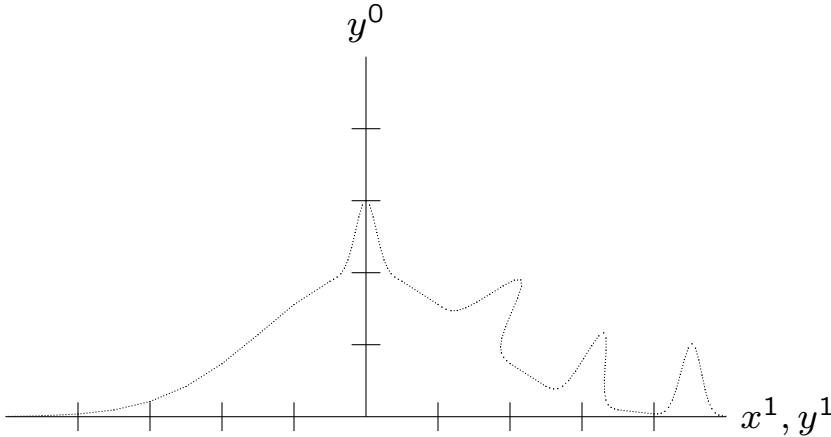
The m th salient main semiaxis can have any direction η_m^r , but usually is either:

- parent's unique normal vector,
- *branch angle*, coplanar with parent's positive main semiaxis,
- fixed angle to rectangular axes y^r .

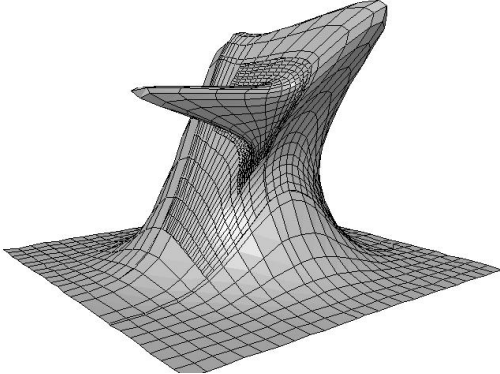
Rule can be an inherited.



Parameter Stretching and Coordinate Curve Obliquity



Child Salients Affected by Parameter Stretching



Coordinate Curve Obliquity

Arc-Length and Oblique Coordinate Transformations

Given by metric tensor g_{ij} at salient origin.

Arc-length coordinate transformation (2D)

$$[\lambda_i^\varepsilon] = \begin{bmatrix} \sqrt{g_{11}} & 0 \\ 0 & \sqrt{g_{22}} \end{bmatrix}.$$

Oblique coordinate transformation (2D)

$$[\omega_\varepsilon^\gamma] = \begin{bmatrix} 1 & \frac{g_{12}}{\sqrt{g_{11}}\sqrt{g_{22}}} \\ 0 & \sqrt{1 - \frac{(g_{12})^2}{g_{11}g_{22}}} \end{bmatrix},$$

from Gram-Schmidt orthonormalization.

Semiaxis Alignment Coordinate Transformation

Semiaxes are aligned with either:

- principal directions at bud, eigenvectors of

$$[b_{i\alpha}] [x^\alpha] = \kappa [g_{i\alpha}] [x^\alpha].$$

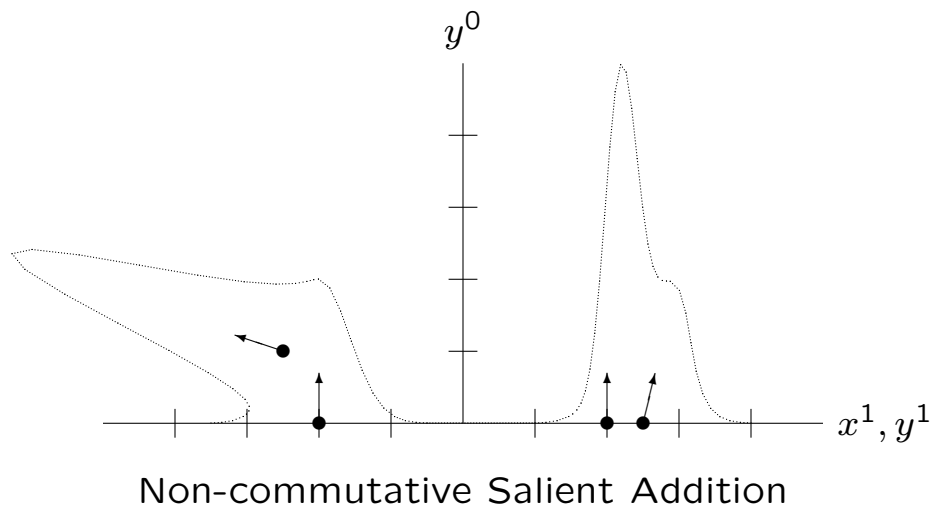
- *branch angle* direction, in the normal section that is parallel to the parent's positive main semiaxis,
- fixed direction relative to rectangular axes y^r ,
- one coordinate curve tangent vector.

Rule can be an inherited.

Salient Addition

Salient addition is closed.

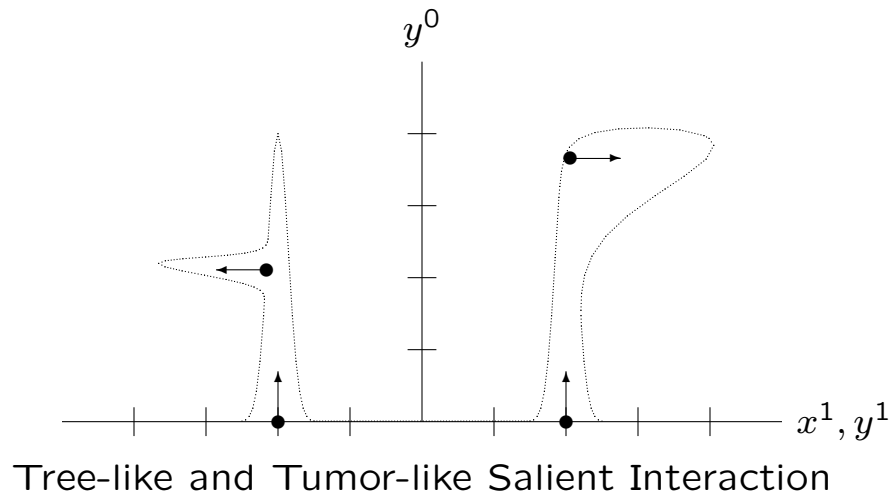
Addition of a coupled salient is non-commutative and non-associative.



Salient Attachment in High-Curvature Regions

Definition 15 *If child salient is smaller, the interaction is **hierarchical** or **tree-like**.*

Definition 16 *If child salient is approximately the same size or larger, the interaction is **tumor-like**, or if flattened **anvil-like**.*



Two widely separated salients m_1 and m_2 are approximately orthogonal,

$$\int_{\mathbb{R}^n} |\hat{S}^{m_1} \hat{S}^{m_2}| dx^1 dx^2 \dots dx^n \approx 0.$$

Salient Assemblage Representation

Overall coordinate transformation

$$\Upsilon_{(m)i}^j \equiv \chi_{(m)\alpha}^j \zeta_{(m)\beta}^\alpha \varsigma_{(m)\gamma}^\beta \omega_{(m)\varepsilon}^\gamma \lambda_{(m)i}^\varepsilon.$$

Canonical coordinates

$$\bar{x}_{(m)}^j = \Upsilon_{(m)i}^j \left(x^i - \dot{X}_{(m)}^i \right).$$

Parametric representation

$$\begin{aligned} y^r &= \delta_i^r x^i + \eta_m^r \hat{S}^m, \\ &= \delta_i^r x^i + \eta_m^r \prod_j c_{(mj)}^{h_j} S_{(j);j}^{h_j} \left(\bar{x}_{(m)}^j \right). \end{aligned}$$

First partial derivative

$$\begin{aligned} y_{,k}^r &= \delta_k^r + \eta_m^r \Upsilon_{(m)k}^\alpha \hat{S}_{,\alpha}^m, \\ &= \delta_k^r + \eta_m^r \Upsilon_{(m)k}^\alpha \prod_j c_{(mj)}^{h_j} S_{(j);j}^{h_j + \delta_j^\alpha} \left(\bar{x}_{(m)}^j \right). \end{aligned}$$

Second partial derivative

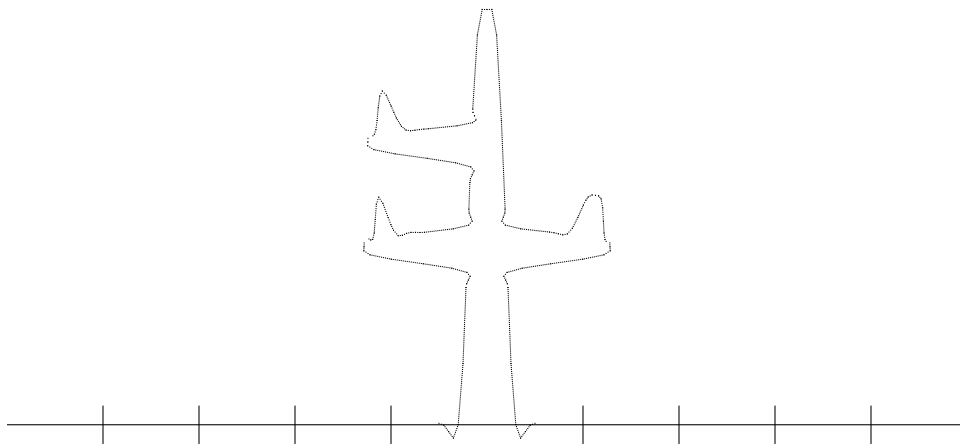
$$\begin{aligned} y_{,kl}^r &= \eta_m^r \Upsilon_{(m)k}^\alpha \Upsilon_{(m)l}^\beta \hat{S}_{,\alpha\beta}^m, \\ &= \eta_m^r \Upsilon_{(m)k}^\alpha \Upsilon_{(m)l}^\beta \prod_j c_{(mj)}^{h_j} S_{(j);j}^{h_j + \delta_j^\alpha + \delta_j^\beta} \left(\bar{x}_{(m)}^j \right). \end{aligned}$$

1D ExpHermite Assemblage

ExpHermite salients in the form

$$\hat{S}^m = \exp^m \left(-(\bar{x}_{(m)}^1)^2 \right) c_{(m)}^h H_h \left(\bar{x}_{(m)}^1 \right).$$

combine to form assemblage like



Cone, Parabola, and Rectangle in Tree

ExpHermite Salient Assemblage Representation

Parametric representation

$$y^r = \delta_i^r x^i + \eta_m^r \exp^m \left(-\bar{x}_{(m)}^\gamma \bar{x}_{(m)}^\gamma \right) \prod_j c_{(mj)}^{h_j} H_{h_j} \left(\bar{x}_{(m)}^j \right).$$

First partial derivative

$$y_{,k}^r = \delta_k^r + \eta_m^r \Upsilon_{(m)k}^\alpha \exp^m \left(-\bar{x}_{(m)}^\gamma \bar{x}_{(m)}^\gamma \right) \prod_j c_{(mj)}^{h_j} H_{h_j + \delta_j^\alpha} \left(\bar{x}_{(m)}^j \right).$$

Second partial derivative

$$y_{,kl}^r = \eta_m^r \Upsilon_{(m)k}^\alpha \Upsilon_{(m)l}^\beta \exp^m \left(-\bar{x}_{(m)}^\gamma \bar{x}_{(m)}^\gamma \right) \prod_j c_{(mj)}^{h_j} H_{h_j + \delta_j^\alpha + \delta_j^\beta} \left(\bar{x}_{(m)}^j \right).$$

Global Cylindrical Coordinates

(ρ, θ, z) to rectangular y^r

$$\begin{aligned}y^0 &= z, \\y^1 &= \rho \cos \theta, \\y^2 &= \rho \sin \theta.\end{aligned}$$

Inverse transformation

$$\begin{aligned}z &= y^0, \\ \rho &= \sqrt{(y^1)^2 + (y^2)^2}, \\ \theta &= \tan^{-1}(y^2/y^1).\end{aligned}$$

Global cylinder $\rho = R$ is

$$\begin{aligned}x^1 &= R \sin \theta, \\x^2 &= z.\end{aligned}$$

Global point $(\dot{\Theta}, \dot{Z})$ becomes m_0 salient origin

$$\begin{aligned}\dot{X}_{(0)}^1 &= R \sin \dot{\Theta}_{(0)}, \\ \dot{X}_{(0)}^2 &= \dot{Z}_{(0)}.\end{aligned}$$

Global Spherical-Polar Coordinates

(ρ, ϕ, θ) to rectangular y^r

$$\begin{aligned}y^0 &= \rho \cos \phi, \\y^1 &= \rho \sin \phi \cos \theta, \\y^2 &= \rho \sin \phi \sin \theta.\end{aligned}$$

Inverse transformation

$$\begin{aligned}\rho &= \sqrt{(y^0)^2 + (y^1)^2 + (y^2)^2}, \\ \phi &= \tan^{-1} \left(\sqrt{(y^1)^2 + (y^2)^2} / y^0 \right), \\ \theta &= \tan^{-1} (y^2 / y^1).\end{aligned}$$

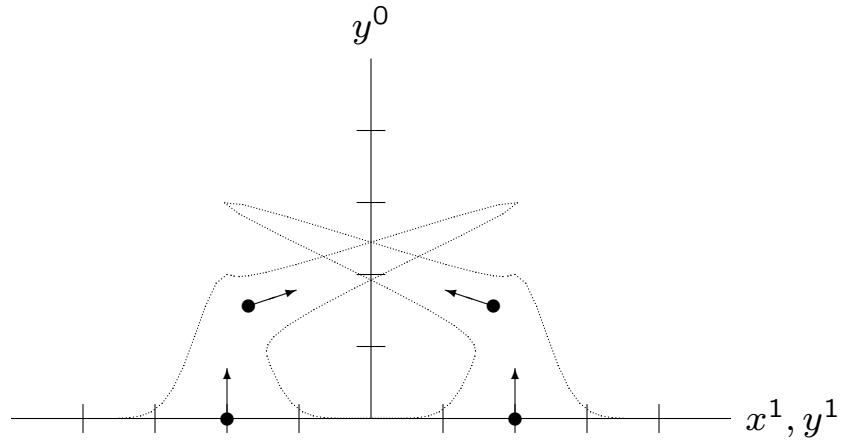
Global sphere $\rho = R$ is

$$\begin{aligned}x^1 &= R \sin \phi \cos \theta, \\x^2 &= R \sin \phi \sin \theta,\end{aligned}$$

Global point $(\dot{\Theta}, \dot{\Phi})$ becomes m_0 salient origin

$$\begin{aligned}\dot{X}_{(0)}^1 &= R \sin \dot{\Phi}_{(0)} \cos \dot{\Theta}_{(0)}, \\ \dot{X}_{(0)}^2 &= R \sin \dot{\Phi}_{(0)} \sin \dot{\Theta}_{(0)}.\end{aligned}$$

Assemblage Self-Intersection



Position vectors y^r of main semi-axes:

$$\begin{aligned} y^{r(m_1)} &= \dot{Y}_{(m_1)}^r + t_1 \eta_{(m_1)}^r, \\ y^{r(m_2)} &= \dot{Y}_{(m_2)}^r + t_2 \eta_{(m_2)}^r, \end{aligned}$$

where $\dot{Y}_{(m_1)}^r$ and $\dot{Y}_{(m_2)}^r$ are buds, and t_1 and t_2 are scalar real parameters. Perpendicular connecting vector

$$\begin{aligned} (y^{r(m_1)} - y^{r(m_2)}) \eta_{r(m_1)} &= 0, \\ (y^{r(m_1)} - y^{r(m_2)}) \eta_{r(m_2)} &= 0. \end{aligned}$$

So

$$\begin{bmatrix} \eta_{(m_1)}^r \eta_{r(m_1)} & -\eta_{(m_2)}^r \eta_{r(m_1)} \\ \eta_{(m_1)}^r \eta_{r(m_2)} & -\eta_{(m_2)}^r \eta_{r(m_2)} \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} -(\dot{Y}_{(m_1)}^r - \dot{Y}_{(m_2)}^r) \eta_{r(m_1)} \\ -(\dot{Y}_{(m_1)}^r - \dot{Y}_{(m_2)}^r) \eta_{r(m_2)} \end{bmatrix}$$

Concise Data Storage

Salient-constant array for 2D assemblage

$$\begin{bmatrix} c_{(0)} & \dot{X}_{(0)}^1 & \bar{X}_{(0)}^{(1)} & \text{shape}_{(0)}^1 & \zeta_{(0)} & \dot{X}_{(0)}^2 & \bar{X}_{(0)}^{(2)} & \text{shape}_{(0)}^2 \\ c_{(1)} & \dot{X}_{(1)}^1 & \bar{X}_{(1)}^{(1)} & \text{shape}_{(1)}^1 & \zeta_{(1)} & \dot{X}_{(1)}^2 & \bar{X}_{(1)}^{(2)} & \text{shape}_{(1)}^2 \\ c_{(2)} & \dot{X}_{(2)}^1 & \bar{X}_{(2)}^{(1)} & \text{shape}_{(2)}^1 & \zeta_{(2)} & \dot{X}_{(2)}^2 & \bar{X}_{(2)}^{(2)} & \text{shape}_{(2)}^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

One row for each salient.

Facilitate telecommunicating a complicated geometry.

Efficient Computation

Assemblage

- Predict negligible terms from parameter values.
- Univariate factors in tensor-product.
- If recursive formula exists and is more efficient, use it.
- For non-deforming, precompute assemblage constants.
- For non-deforming, precompute coupling matrix.
- For partial derivatives, reuse previously computed function evaluations and repeating chain-rule factors.

ExpHermite Assemblage

- Exponent function $\exp(-x^2)$ factors out, leaving efficient polynomial.
- Transform Hermite series to power series.
- If factor is even or odd function, half the terms are zero and can be bypassed.

Transform Hermite Series to Power Series

Hermite series has equivalent power series

$$d_{(mj)\delta_j^\alpha h_k}^{q_j} P_{q_j}(\bar{x}_{(m)}^j) = c_{(mj)}^{h_j} H_{h_j + \delta_j^\alpha h_k}(\bar{x}_{(m)}^j),$$

where

$$P_{q_j}(\bar{x}_{(m)}^j) \equiv (\bar{x}_{(m)}^j)^{q_j}.$$

Coefficients transform as

$$d_{(mj)h_k}^{q_j} = \varpi_{\vartheta}^{q_j} \delta_{\theta + h_k}^{\vartheta} c_{(mj)}^{\theta},$$

where

$$\left[\varpi_{\vartheta}^{q_j} \right] = \begin{bmatrix} 1 & 0 & -2 & 0 & 12 & & \\ 0 & -2 & 0 & 12 & 0 & & \\ 0 & 0 & 4 & 0 & -48 & & \\ 0 & 0 & 0 & -8 & 0 & & \\ 0 & 0 & 0 & 0 & 16 & & \\ & & & & & \dots & \end{bmatrix}.$$

Multiplication $\delta_{\theta + h_k}^{\vartheta} c_{(mj)}^{\theta}$ is equivalent to a shift of array c elements.

Differential Geometry

Parametric representation and its first two partial derivatives

$$y^r, y^r_{,k}, y^r_{,kl}.$$

Jacobian matrix

$$J \equiv [J^r_i]_{(n+1) \times n} \equiv y^r_{,i}.$$

Base vectors are functions of position (curvilinear coordinates)

$$\mathbf{a}_i = \mathbf{y}_{,i} \equiv y^{\rho}_{,i} \mathbf{e}_{\rho}.$$

Metric tensor

$$g_{ij} \equiv \mathbf{a}_i \cdot \mathbf{a}_j.$$

Since y^r are rectangular coordinates with the *Euclidean metric*,

$$[g_{ij}] = J^T J = [y^{\rho}_{,i} y^{\rho}_{,j}].$$

Differential Geometry

A more general *Riemannian metric*, including non-Euclidean, is

$$[g_{ij}] = J^T G J.$$

If J is full rank, then there exists a g^{ij} such that

$$g^{i\alpha} g_{\alpha j} = \delta_j^i.$$

Metric tensor determinant

$$g \equiv |g_{ij}|.$$

For two dimensions, $g = g_{11}g_{22} - (g_{12})^2$.

Cosine between x^i and x^j -parametric curves

$$\cos \omega = g_{ij} / \sqrt{g_{ii} g_{jj}} \quad (\text{no sum on } i, j).$$

Invariant *First Fundamental Form*

$$I \equiv g_{ij} dx^i dx^j.$$

Differential Geometry

Christoffel symbols of the first kind

$$\Gamma_{ijk} \equiv \frac{1}{2} (g_{jk,i} + g_{ki,j} - g_{ij,k}).$$

With rectangular coordinates y^r and Euclidean metric

$$\Gamma_{ijk} = y_{,k}^{\rho} y_{,ij}^{\rho}.$$

Christoffel symbols of the second kind

$$\Gamma_{ij}^k \equiv g^{k\alpha} \Gamma_{ij\alpha}.$$

Riemannian tensor of the second kind

$$R_{jkl}^i \equiv \Gamma_{jl,k}^i - \Gamma_{jk,l}^i + \Gamma_{jl}^{\alpha} \Gamma_{\alpha k}^i - \Gamma_{jk}^{\alpha} \Gamma_{\alpha l}^i.$$

Riemannian tensor of the first kind

$$R_{ijkl} \equiv g_{i\alpha} R_{jkl}^{\alpha}.$$

Gaussian curvature on a two-dimensional surface

$$K = \frac{R_{1212}}{g_{11}g_{22} - (g_{12})^2}.$$

Differential Geometry

Surface curve, function of parameter t ,

$$x^i \equiv x^i(t),$$

in ambient coordinates y^r , is composite function

$$y^r \equiv y^r(x^i(t)).$$

Curve's tangent is given by chain-rule

$$\frac{dy^r}{dt} = y^r_{,i} \frac{dx^i}{dt}.$$

Square of differential arc length is

$$(ds)^2 = g_{ij} dx^i dx^j.$$

Curve arc length is

$$s = \int_{x_0}^{x_1} \sqrt{g_{ij} dx^i dx^j} = \int_{t_0}^{t_1} \sqrt{g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}} dt.$$

Two-dimensional surface area is

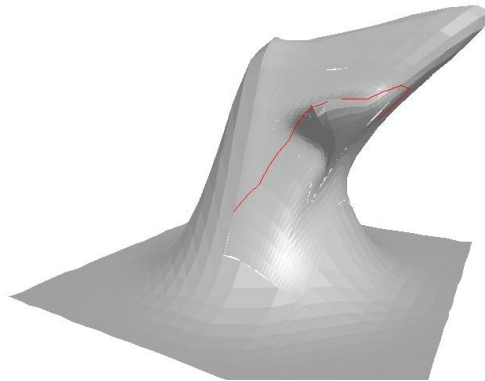
$$A = \int_{\mathcal{X}} \sqrt{g} dx^1 dx^2.$$

Geodesic

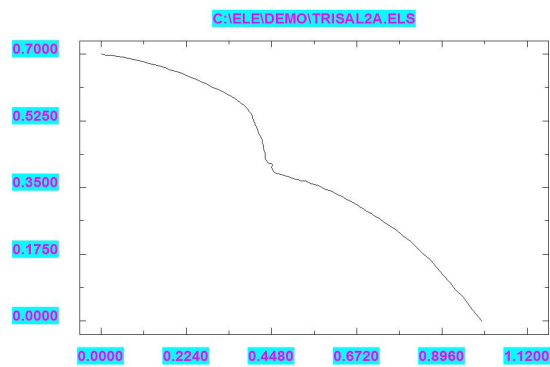
A *geodesic* $x^i(s)$ solves system of differential equations

$$\frac{d^2x^k}{ds^2} + \Gamma_{ij}^k \frac{dx^i}{ds} \frac{dx^j}{ds} = 0.$$

Geodesic on surface (<http://www.netlib.org/ode/geodesic/>)



Geodesic in parameter space



Differential Geometry

Surface normal vector

$$N_r = \epsilon_{rst} y_{,1}^s y_{,2}^t.$$

Unit normal vector

$$n_r \equiv N_r / \sqrt{N_\rho N_\rho}.$$

Invariant *Second Fundamental Form*

$$II \equiv b_{ij} dx^i dx^j,$$

with coefficients from *curvature tensor*

$$b_{ij} \equiv y_{,ij}^\rho n_\rho.$$

For two dimensions,

$$b \equiv |b_{ij}| = b_{11}b_{22} - (b_{12})^2.$$

Gauss equation

$$y_{,ij}^r = \Gamma_{ij}^\alpha y_{,\alpha}^r + b_{ij} n^r.$$

Weingarten equation

$$n_{,i}^r = -g^{\alpha\beta} b_{i\alpha} y_{,\beta}^r = -b_i^\beta y_{,\beta}^r.$$

Tensor Applications

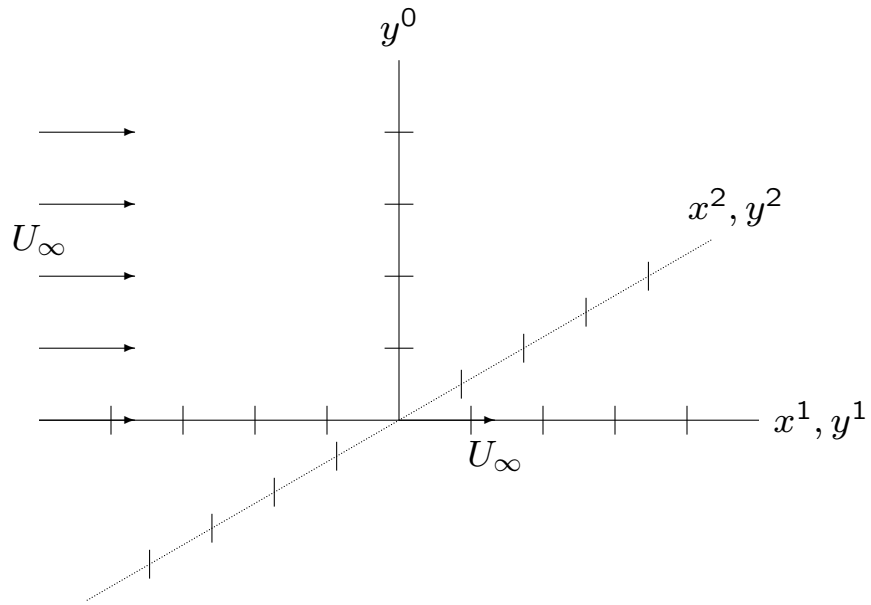
Newton's second law

$$\begin{aligned} F^r &= m \frac{dv^r}{dt}, \\ &= m \left(\frac{d^2 y^r}{dt^2} + \Gamma_{ij}^r \frac{dx^i}{dt} \frac{dx^j}{dt} \right), \end{aligned}$$

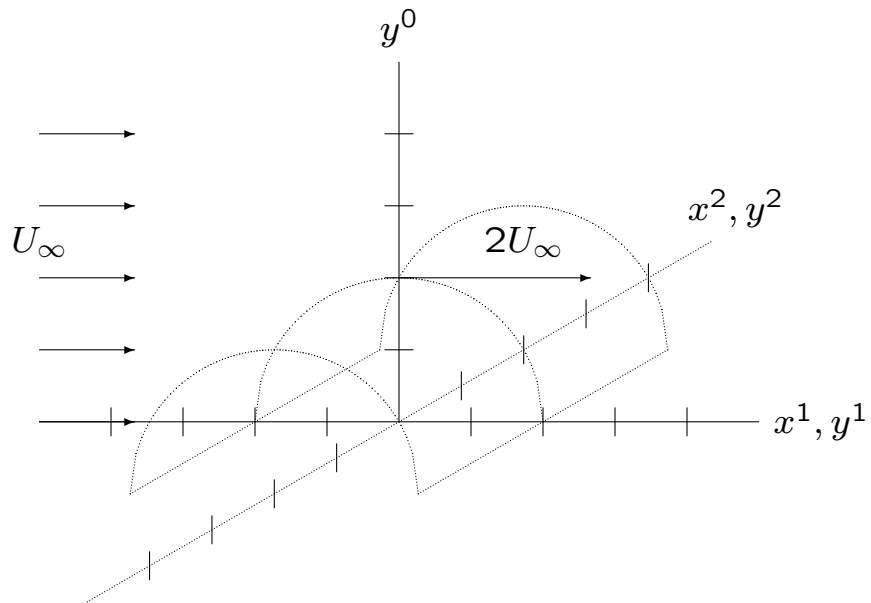
is valid in all coordinate systems.

Describes a force field on a curved surface, like the interface between two fluids.

Potential Flow Examples

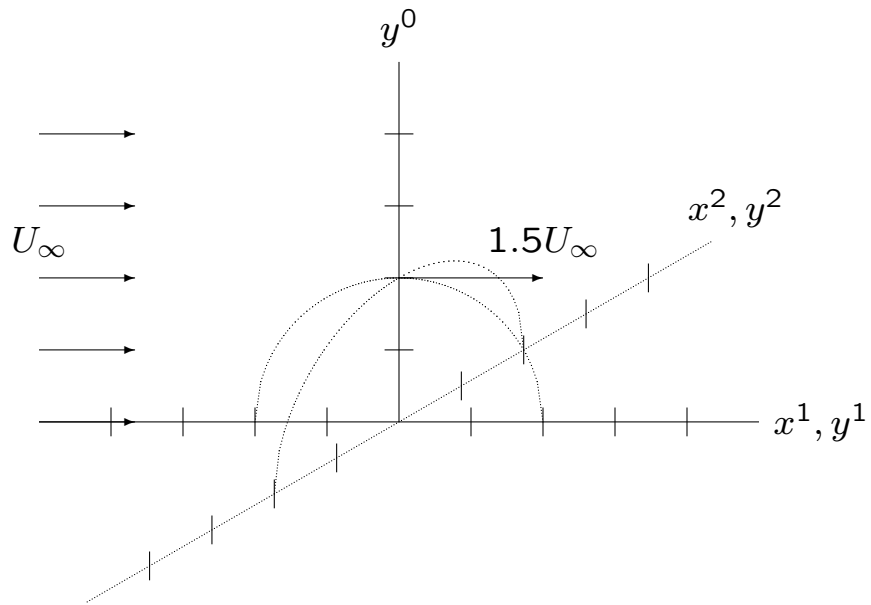


Potential Flow over Plane: $\phi = -U_\infty x^2$

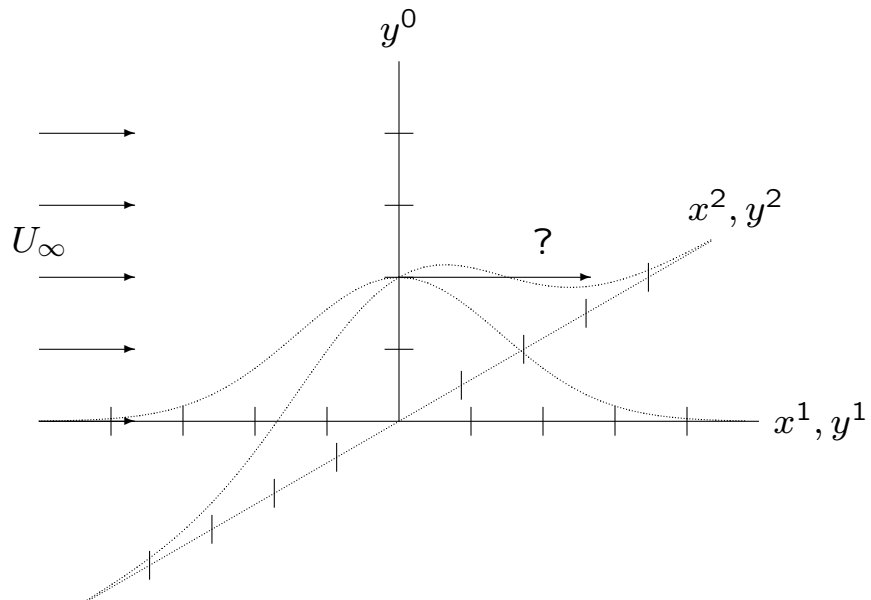


Potential Flow over Cylinder: $\phi = -U_\infty \left(1 + \frac{R^2}{r^2}\right) r \cos \theta$

Potential Flow Examples



Potential Flow over Sphere: $\phi = -U_\infty \left(1 + \frac{R^3}{2r^3}\right) r \cos \theta$



Potential Flow over Salient

Conclusions

Salient assemblage representation is important because:

- Complexity of many problems stems from representation of irregular or deforming geometry. An assemblage decomposes a geometric object into asymptotic blending salient units. It models a multidimensional parametric system.
- A linear combination of a 1D salient and its derivatives spans a wider collection of 1D salients.
- Salients are attached with recursive rules on dihedral and semiaxis alignment.
- An assemblage covers the surface of interest with one patch.
- An assemblage has concise data storage.
- It allows efficient computation.

ExpHermite assemblage representation has advantages:

- Built-in data fitting using ExpHermite series, a generalized Fourier series.
- Efficient polynomial computation.